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Self-Gravitating Three-Dimensional Solitons in Nonlinear Scale-Invariant Electrodynamics

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A scale-invariant nonlinear modification of Maxwellian electrodynamics within general relativity is proposed. The starting point is the Mie model and its scale-invariant generalization in flat space-time E_4. We prove that all static, spherically symmetrical regular field configurations in this new theory, as well as those in the Mie model, possess negative energy. In search of solitonlike solutions with positive masses, we take into account their proper gravitational fields. We show first that in Riemannian space any gauge-invariant electrodynamic theory does not admit regular solutions. Supposing the gauge invariance to be broken inside the particle, we prove the existence of static particlelike solutions with spherical symmetry and positive energy in the scale-invariant electrodynamics described by a Lagrangian density of the form $\mathcal{L} = -Y(I)R/(2\kappa) - W(I)F_{\alpha\beta}F^{\alpha\beta}/2 + 2X(I)R_{\alpha\beta}A^{\alpha}A^{\beta}$, with Y, W, and X arbitrary functions of the invariant $I = A_{\alpha}A^{\alpha}$. The correspondence with the Maxwellian theory is required.

1. INTRODUCTION

The first nonlinear model of electrodynamics generalizing the system of Maxwell's equations was suggested by Mie (1912), who introduced a unified field picture of matter supposing the density of the current to be a function of electromagnetic potentials. This conjecture led him to a Lagrangian density of the form

$$\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu} + g(A_{\alpha}A^{\alpha})^3$$

with $g < 0$ being an arbitrary constant, the negative sign of which was necessary for the existence of regular solutions.

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To justify the fact that the Mie model contains explicitly the 4-potential A_μ, which in the case of Maxwellian electrodynamics is defined up to a gradient transformation, one can notice that the gauge invariance is broken only inside the particles, where the invariant $I = A_\mu A^\mu$ is large. However, the gauge invariance is reinstated in the weak-field limit corresponding to the Maxwellian theory. It turns out that regular solutions of the Mie model possess negative energy; therefore it needs to be modified in order to cover particles of positive mass.

This paper is arranged as follows. In Section 2 we construct an extension of the Mie model on the basis of scale-invariant electrodynamics in flat space-time E_4. We prove that all static, spherically symmetrical regular solutions in this new theory, as well as those in the Mie model, describe field configurations with negative energy. To meet the requirement of mass positivity we take into account the interaction between electromagnetic and gravitational fields.

In Section 3 we show that any model of gauge-invariant electrodynamics in Riemannian space does not possess solitonlike configurations. Then we construct the Lagrangian of the scale-invariant electrodynamics with broken gauge invariance inside the particle and prove the existence of regular static, spherically symmetrical solutions with positive energy. In this connection it should be emphasized that by only taking into account the proper gravitational field of the particle can one succeed in searching for regular solutions with positive mass, this result being impossible in Minkowski space.

2. EXTENSION OF MIE'S MODEL ON THE BASIS OF SCALE-INVARIANT ELECTRODYNAMICS IN FLAT SPACE-TIME

In this paper we consider the simplest type of three-dimensional solitons—static, spherically symmetrical ones. In this section we first construct a scale-invariant electrodynamic theory in flat space-time E_4. One can easily prove that gauge-invariant electrodynamics does not admit regular solutions in Minkowski space due to the divergence form of the field equations (Bronnikov, 1992). Thus, if one assumes the differential equations to be of second order, it seems natural to consider the Lagrangian as a function of at least two invariants,

$$I_M = -\tfrac{1}{2}F_{\mu\nu}F^{\mu\nu}, \qquad I = A_\mu A^\mu$$

just supposing the gauge invariance to be broken. We also require that at large distance from the particle the correspondence principle with linear Maxwell electrodynamics holds. Moreover, we suppose that the new model preserves scale invariance, which is one of the attractive inherent properties of Maxwell's theory. Namely, noticing that in static electrodynamics only the time component $A_0 = \varphi$ survives, let us perform the following transformations of the radial variable r and the scalar potential $\varphi(r)$:

$$r \to ar, \qquad \varphi(r) \to b\varphi(r)$$

with a and b arbitrary constant parameters. Then the invariance of the action under these transformations implies that the Lagrangian density $\mathcal{L}$ satisfies an equation of the form

$$\mathcal{L}(I_M, I) = a^3\mathcal{L}\left(\frac{b^2}{a^2} I_M, b^2 I\right) \tag{2.1}$$

The correspondence with the Maxwellian theory in the limit $I \to 0$ permits one to deduce from (2.1) the following relation between the parameters a and b:

$$ab^2 = 1 \tag{2.2}$$

In view of (2.1) and (2.2) the Lagrangian density becomes

$$\mathcal{L} = I_M U\left(\frac{I^3}{I_M}\right) \tag{2.3}$$

where U is an arbitrary function such that $U(0) = 1/8\pi$.

One gets from (2.3) the following Euler–Lagrange equation:

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\varphi'\left(U - \frac{\varphi^6}{\varphi'^2}U_\eta\right)\right] - 3\varphi^5 U_\eta = 0 \tag{2.4}$$

where U_η denotes the derivative of the function U with respect to its argument $\eta = \varphi^5/\varphi'^2$. The dilatation invariance of (2.4) permits one to apply Noether's theorem and obtain the integral of motion

$$\frac{\varphi^6}{\varphi'^2}(\varphi + 2r\varphi')U_\eta - (\varphi + r\varphi')U = 0 \tag{2.5}$$

where the integration constant vanishes due to the correspondence with Maxwell's theory. Integrating (2.5), one finds the following parametrization of the curve $\varphi = \varphi(r)$:

$$r(\eta) = \frac{8\pi q^2}{\sqrt{\eta}}(U - \eta U_\eta), \qquad U - \eta U_\eta \geq 0 \tag{2.6}$$

$$\varphi^2(\eta) = \frac{\eta}{8\pi q^2}\frac{1}{U - 2\eta U_\eta}, \qquad U - 2\eta U_\eta > 0$$

where q stands for the electrical charge of the particle.

Let us investigate the behavior of solutions (2.6) in the neighborhood of the center of the system. Regularity requirements (at $r = r_c = 0$, $\varphi = \varphi_c < \infty$, $\varphi' = 0$) impose the following restrictions on the function U:

$$\lim_{\eta\to\infty} \frac{\eta}{U - 2\eta U_\eta} < \infty, \qquad \lim_{\eta\to\infty} \frac{U - \eta U_\eta}{\sqrt{\eta}} = 0$$

which are met if and only if at large η

$$U \approx g\eta + O(1), \qquad g < 0 \tag{2.7}$$

where the negative sign of the constant g is chosen due to the positivity of $U - 2\eta U_\eta$. Notice that the derivative

$$\varphi' = -\eta[8\pi q^2(U - 2\eta U_\eta)]^{-3/2}$$

in view of (2.7) automatically vanishes at the center. It should be noted that the Mie model corresponds to the choice $U = 1/(8\pi) + g\eta$.

Now let us estimate the energy of the field configuration found:

$$E = \int T_0^0\, dV = 4\pi \int_0^\infty (\varphi'^2 U + 4\varphi^6 U_\eta) r^2\, dr$$

where the symmetrical energy-momentum tensor was utilized. Using the integral identity

$$\varphi\varphi'(U - \eta U_\eta) r^2 |_0^\infty - \int_0^\infty \varphi'^2 (U - \eta U_\eta) r^2\, dr = 3 \int_0^\infty \varphi^6 U_\eta r^2\, dr$$

which can be obtained if both sides of (2.4) are multiplied by φ and integrated over the space, one gets

$$E = 8\pi \int_0^\infty \varphi^6 U_\eta r^2\, dr = -\frac{8\pi}{3} \int_0^\infty \varphi'^2 (U - \eta U_\eta) r^2\, dr \tag{2.8}$$

As follows from (2.6) and (2.8), the field energy proves to be negative. Thus we conclude that in flat space-time a regular scale-invariant electrodynamics of the Mie type can be constructed only for particles with negative mass.

3. MODEL OF SCALE-INVARIANT ELECTRODYNAMICS IN GENERAL RELATIVITY

In this section, with the aim to describe field configurations with positive energy, we take into account the interaction between electromagnetic and gravitational fields. We formulate now the regularity criteria for the self-consistent configurations of electromagnetic and gravitational fields with regular center. Regularity means that in the neighborhood of the center the following conditions are satisfied (Bronnikov *et al.*, 1979):

(1) All metric components and electromagnetic potentials are regular.
(2) The space is Euclidean.
(3) Electromagnetic and gravitational forces vanish at the center.

We also require that asymptotically, at large distances from the center, the Maxwell–Einstein equations are valid.

Let us first study a generalization of the so-called Born–Infeld gauge-invariant electrodynamics (Born, 1934) described by a Lagrangian density of the form

$$\mathcal{L} = \frac{R}{2\kappa} + X(I_M)$$

where R is the Riemannian scalar curvature, κ is the Einstein gravitational constant, and X is an arbitrary function such that in the limit of weak fields $X \to I_M/(8\pi)$.

Restricting ourselves to static, spherically symmetrical fields, we can write the squared interval as

$$ds^2 = e^{2\gamma(r)}\,dt^2 - e^{2\alpha(r)}\,dr^2 - r^2\,d\Omega^2$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\,d\varphi^2$ and the functions α and γ satisfy the following Einstein equations:

$$e^{-2\alpha}\left(\frac{1}{r^2} - \frac{2\alpha'}{r}\right) - \frac{1}{r^2} = -\kappa T_0^0 \tag{3.1}$$

$$e^{-2\alpha}\left(\frac{1}{r^2} + \frac{2\gamma'}{r}\right) - \frac{1}{r^2} = -\kappa T_1^1 \tag{3.2}$$

$$e^{-2\alpha}\left(\gamma'' + \gamma'^2 + \frac{1}{r}(\gamma' - \alpha') - \gamma'\alpha'\right) = -\kappa T_2^2 = -\kappa T_3^3$$

Here $T_0^0 = T_1^1 = 2X_{I_M} - X$, $T_2^2 = T_3^3 = -X$ are the components of the electromagnetic energy-momentum tensor, with X_{I_M} denoting dX/dI_M.

The scalar potential φ obeys Maxwell's equation $(e^{-\alpha-\gamma}r^2\varphi' X_{I_M})' = 0$, with the evident solution

$$X_{I_M}\varphi' = -\frac{1}{8\pi}\frac{q}{r^2} \tag{3.3}$$

Here we have taken into account the relation $\alpha + \gamma = 0$ which follows from equation (3.2) minus equation (3.1). One concludes from (3.3) that the integration constant q is the electrical charge of the particle.

As is clear from (3.3), the electrostatic force vanishes at the center $r = 0$ ($\varphi' = 0$) if and only if X_{I_M} becomes infinite at least as r^{-3}. But the latter behavior is impossible due to the fact that in the limit of weak fields ($\varphi'^2 \to 0$), $X_{I_M} \to 1/(8\pi)$, as was previously assumed. Therefore, we come to the conclusion that gauge-invariant electrodynamics in general relativity does not admit regular static, spherically symmetrical configurations.

Thus, to find a model of regular electrodynamics, we come to the necessity of breaking the gauge invariance by including terms in the Lagrangian with explicit dependence on electromagnetic potentials.

Consider the Lagrangian for the electromagnetic and gravitational fields as a functional of four independent invariants (Chugunov *et al.*, 1994):

$$L = \int_{(V)} \mathcal{L}(R, I_M, R_{\alpha\beta}A^\alpha A^\beta, I)\sqrt{-g}\, d^3x$$

The $SO(3)$ invariance of this Lagrangian permits one to apply the Palais symmetric criticality principle (Palais, 1979) and to take the 4-potential and the squared interval in the form

$$A_\mu = A(\xi)\delta^0_\mu$$
$$ds^2 = e^{2\gamma(\xi)}\, dt^2 - e^{2\alpha(\xi)}\, d\xi^2 - e^{2\beta(\xi)}\, d\Omega^2$$

where $\xi = \xi(r)$ is the auxiliary coordinate corresponding to $\xi = 1/r$ in the flat space-time E_4 due to the following constraint:

$$\gamma = \alpha + 2\beta. \tag{3.4}$$

Using the fact that the symmetry properties of field equations facilitate the search for their solutions, let L show invariance under the following scale transformations:

$$e^{2\gamma(\xi)} \to k_1 e^{2\gamma(\xi)}, \qquad e^{2\alpha(\xi)} \to k_2 e^{2\alpha(\xi)}, \qquad e^{2\beta(\xi)} \to k_3 e^{2\beta(\xi)} \tag{3.5}$$
$$\xi \to a\xi, \qquad A(\xi) \to bA(\xi)$$

with k_i, a, and b being group parameters. It should be noted that these parameters are not all independent. Indeed, for the scalar curvature $R = 2(R^{01}_{01} + 2R^{02}_{02} + 2R^{12}_{12} + R^{23}_{23})$ to be pulled to R/k_3 by the transformations (3.5), it is necessary to put $k_3 = a^2 k_2$ in view of the fact that

$$R^{0i}_{0i} \to R^{0i}_{0i}/(k_2 a^2), \qquad R^{12}_{12} \to R^{12}_{12}/(k_2 a^2), \qquad i = 1,2$$
$$R^{23}_{23} = e^{-2\beta} - \beta'^2 e^{-2\alpha} \to e^{-2\beta}/k_3 - \beta'^2 e^{-2\alpha}/(a^2 k_2)$$

Furthermore, the correspondence with the Maxwell–Einstein theory, that is, the behavior $\mathcal{L} \to R/(2\kappa) + I_M/(8\pi)$ in the limit $I \to 0$, implies two other relations: $k_1k_3 = 1$, $k_1 = b^2$. Lastly, from (3.4) we get $k_3 = k_2$. Therefore the invariance of the Lagrangian L implies a functional equation of the form

$$\mathcal{L}(k_1 R, k_1 I_M, k_1 R_{\alpha\beta}A^\alpha A^\beta, I) = k_1\mathcal{L}(R, I_M, R_{\alpha\beta}A^\alpha A^\beta, I)$$

whence the structure of the Lagrangian density $\mathcal{L}$ immediately follows:

$$\mathcal{L} = -Y(I)\frac{R}{2\kappa} - \frac{1}{2}W(I)F_{\alpha\beta}F^{\alpha\beta} + 2X(I)R_{\alpha\beta}A^\alpha A^\beta \tag{3.6}$$

where Y, W, and X are arbitrary functions of the invariant I such that in the limit $I \to 0$, $Y \to 1$, $W \to 1/(8\pi)$, and $IX \to 0$.

Using (3.6), one gets the total Lagrangian in the form

$$\begin{aligned} L = -\frac{1}{\kappa}\int_0^\infty & e^{-\alpha+\gamma+2\beta}[(3\beta'^2 - 2\beta'\alpha' + 2\beta'' - e^{2\alpha-2\beta})Y(I) \\ & + (\gamma'^2 + \gamma'' - \alpha'\gamma' + 2\gamma'\beta')F(I)]\,d\xi + \int_0^\infty A'^2 e^{-\alpha-\gamma+2\beta}W(I)\,d\xi \end{aligned} \tag{3.7}$$

with the notation $F = Y - 2\kappa IX$. One easily obtains from (3.7) the following Euler–Lagrange equations:

$$\begin{aligned} &(Y - 2IY_I)(2\beta'' - \beta'^2 - 2\gamma'\beta' - e^{\alpha+\gamma}) \\ &\quad + F'' - \gamma'F' - 2IF_I\gamma'' = -\kappa A'^2 e^{-2\gamma}(W + 2IW_I) \end{aligned} \tag{3.8}$$

$$\begin{aligned} &Y(\beta'' + \gamma'' - \beta'^2 - 2\beta'\gamma') - \gamma'F' \\ &\quad - Y'(\beta' - \gamma') + Y'' = \kappa A'^2 e^{-2\gamma}W \end{aligned} \tag{3.9}$$

$$Y(\beta'^2 + 2\gamma'\beta') - Ye^{\alpha+\gamma} + \gamma'F' + 2\beta'Y' = -\kappa A'^2 e^{-2\gamma}W \tag{3.10}$$

$$\begin{aligned} &Ae^{-2\gamma}[\gamma''F_I + (2\beta'' - \beta'^2 - 2\gamma'\beta' - e^{2\gamma+2\beta})Y_I] \\ &\quad = \kappa[AA'^2W_I e^{-4\gamma} - (A'e^{-2\gamma}W)'] \end{aligned} \tag{3.11}$$

where the derivative with respect to ξ is denoted by a prime.

The scale invariance of the Lagrangian (3.7) leads to the first integral

$$Q' + \gamma'Y - \kappa AA'e^{-2\gamma}W = N = \text{const} \tag{3.12}$$

where we put $Q = Y - F/2$. It should be mentioned that the integral of motion (3.12) also could be obtained by integrating the sum of equations (3.8) and (3.9) and taking into account (3.11).

Furthermore, one easily sees that the sum of equations (3.9) and (3.10) leads to

$$Y(\beta'' + \gamma'') + Y'(\gamma' + \beta') + Y'' = Ye^{2\gamma+2\beta} \tag{3.13}$$

whence

$$(Y' + Y(\gamma' + \beta'))^2 = Y^2 e^{2(\gamma+\beta)} + k^2 \operatorname{sign} k, \qquad k = \text{const} \tag{3.14}$$

or

$$e^{-\gamma-\beta} = \begin{cases} Y \displaystyle\int_0^\xi Y^{-1}\, d\xi, & k = 0 \\ k^{-1} Y \sin\left(k \displaystyle\int_0^\xi Y^{-1}\, d\xi\right), & k < 0 \\ k^{-1} Y \operatorname{sh}\left(k \displaystyle\int_0^\xi Y^{-1}\, d\xi\right), & k > 0 \end{cases} \tag{3.15}$$

Then (3.9), in view of (3.13) and (3.14), can be rewritten as

$$Y'^2 + Y^2\gamma'^2 - k^2 \operatorname{sign} k + 2YQ'\gamma' = \kappa A'^2 e^{-2\gamma} WY \tag{3.16}$$

Denoting $B(\xi) = Ae^{-\gamma}$, one derives from (3.16) and (3.12) the following pair of differential equations:

$$Y\gamma' = \frac{B(Y'^2 - k^2 \operatorname{sign} k) - YB'(Q' - N)}{YB' - B(N + Q')} \tag{3.17}$$

$$B'^2 = \frac{YN^2 - k^2 \operatorname{sign} k(Y - \kappa WB^2)}{\kappa W(B^2 Y_B^2 - 2BYQ_B + Y^2) + Y(Q_B^2 - Y_B^2)} \tag{3.18}$$

Equation (3.18) can be integrated, as its right-hand side is a function of B. The substitution of the solution $B = B(\xi)$ into (3.17) permits one to find $\gamma = \gamma(\xi)$. The other unknown quantities α and β can be determined via (3.4) and (3.15).

Thus we have shown that, given the functions Y, W, and X, the self-consistent system of gravitational and electromagnetic fields is governed by the integrable equations (3.8)–(3.11).

We analyze now the regularity of the obtained solutions in the neighborhood of the center. In the case of static, spherically symmetrical field configurations the regularity criteria (1)–(3) are reformulated as follows. At some $\xi = \xi_c$ such that $e^{\beta(\xi_c)} = 0$:

(1′) $e^\gamma < \infty$, $e^\alpha < \infty$, $|A| e^{-\gamma} < \infty$.
(2′) $e^{\beta-\alpha}|\beta'| = 1$.
(3′) $A' e^{-\alpha-\gamma} = 0$, $\gamma' e^{-\alpha+\gamma} = 0$.

With the aim to illustrate that the field equations (3.8)–(3.11) possess regular solutions, choose $F \equiv 1$, which leads to the following relations:

$$(Y\gamma' + Y')^2 = \kappa WYA'^2 e^{-2\gamma} + k^2 \operatorname{sign} k \tag{3.19}$$

$$Y\gamma' + Y' = \kappa WAA' e^{-2\gamma} + N \tag{3.20}$$

Putting $k = 0$ to meet the requirements (1′)–(3′), we obtain

$$B'^2 = \frac{N^2 Y}{\kappa W(Y - BY_B)^2} \tag{3.21}$$

$$Y\gamma' = \frac{NYB' - Y'(YB' - Y'B)}{YB' - Y'B - NB} \tag{3.22}$$

Let us now choose for simplicity the functions Y and W as follows:

$$Y = g(B/B_c) + \sqrt{1 - (B/B_c)^2}$$

$$8\pi W = \frac{Y}{1 - (B/B_c)^2}, \qquad g > 0, \qquad B_c = \text{const}$$

Substituting W and Y into (3.21), one derives

$$B = B_c \operatorname{th}(n\xi), \qquad n = q/B_c > 0 \tag{3.23}$$

where g is the electrical charge defined by the boundary condition $B'(0) = q$.

In view of (3.23), equation (3.22) can be easily integrated:

$$e^{(1+\alpha g)\gamma} = \frac{\alpha \operatorname{ch}(n\xi)}{\alpha - \operatorname{sh}(n\xi)} \left[\frac{\operatorname{ch}(n\xi)}{1 + g \operatorname{sh}(n\xi)}\right]^{\alpha g}, \qquad \alpha = \frac{n}{N}$$

The function $\beta(\xi)$ can be found after integrating equation (3.15) under the condition that $k = 0$:

$$e^{-\gamma-\beta} = \frac{1 + g \operatorname{sh}(n\xi)}{ng \operatorname{ch}(n\xi)} \ln[1 + g \operatorname{sh}(n\xi)]$$

Considering (3.19) and (3.20) at $\xi = 0$, one finds the following relations between the model parameters and integration constants:

$$N^2 = Gq^2, \qquad -Gm + gn = N \tag{3.24}$$

where $m = -\gamma'(0)/G$ is the Schwarzschild mass determining the total field energy and $G = 8\pi\kappa$ is the Newtonian gravitational constant. Notice that by specifying the constants in (3.24), one can obtain a positive mass spectrum.

Thus we conclude that a nonlinear scale-invariant model of asymptotically flat electrodynamics admitting particlelike regular configurations with positive energy can be constructed if both the gauge invariance is broken inside the particle and its proper gravitational field is taken into account.

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